

Trace anomalies and chiral Ward identities

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In a simple abelian spinor field theory, the canonical trace identities for certain axial-vector and axial-scalar operators are reexamined in dimensional regularization, some disagreements with previous results are found and an interesting new phenomenon is observed and briefly discussed.

It is well known that chiral anomaly has direct physical and topological connections [1] and similarly for trace anomaly [2–4], such anomalies are often termed as quantum mechanical violation of classical symmetries, specifically, via the regularization effects in quantum field theories. That is, the quantization procedure is incompatible with such symmetries. In 't Hooft's seminal interpretation, chiral anomaly has also been shown to arise from the decoupling of heavy fermions [5], namely, chiral anomaly is closely related to dynamical contents. Therefore, chiral and other anomalies have become the key concern for model construction both in field theories [6] and string theories [7]. Thus anomalies in canonical relations are very important in field theories and high energy physics, their appearances are often helpful in deepening our understanding of the quantum theories.

In this letter, we examine the trace anomalies with an emphasis on the relation between the trace and the chiral identities for certain axial operators, as they are important in modern particle physics, especially in the supersymmetric field theories [8]. Specifically, we examine the trace and chiral relations satisfied by the two- and three- point functions of operators $j_\mu^5 \equiv \bar{\psi}\gamma_\mu\gamma_5\psi$, $j^5 \equiv 2im\bar{\psi}\gamma_5\psi$, $\theta \equiv m\bar{\psi}\psi$, and $\sigma \equiv 4m\bar{\psi}\psi$ computed in dimensional regularization. The non-abelian ones have been examined in Ref. [4] through partial calculation. Here we carry out all the one loop calculations which are in fact very simple and then examine explicitly the relations satisfied by these quantities.

The objects to be calculated are listed as follows,

$$\Pi_{\mu\nu}^5(p, -p) \equiv i\mathcal{FT}\{\langle 0|T(j_{5\mu}j_{5\nu})|0\rangle\}, \quad (1)$$

$$\Delta_{\mu\nu}^5(0, p, -p) \equiv \mathcal{FT}\{\langle 0|T(\theta j_{5\mu}j_{5\nu})|0\rangle\}; \quad (2)$$

$$\Pi_\nu^5(p, -p) \equiv i\mathcal{FT}\{\langle 0|T(j_5j_{5\nu})|0\rangle\}, \quad (3)$$

$$\Delta_\nu^5(0, p, -p) \equiv \mathcal{FT}\{\langle 0|T(\theta j_5j_{5\nu})|0\rangle\}; \quad (4)$$

$$\Pi^5(p, -p) \equiv i\mathcal{FT}\{\langle 0|T(j_5j_5)|0\rangle\}, \quad (5)$$

$$\Delta^5(0, p, -p) \equiv \mathcal{FT}\{\langle 0|T(\theta j_5j_5)|0\rangle\}; \quad (6)$$

$$\langle \sigma \rangle \equiv \mathcal{FT}\{4m\langle \bar{\psi}\psi \rangle\}, \quad (7)$$

$$\Pi^{\theta\sigma}(0, 0) \equiv -i\mathcal{FT}\{\langle 0|\theta\sigma|0\rangle\}, \quad (8)$$

where $\mathcal{FT}\{\dots\}$ denotes the Fourier transform and m refers to the fermion mass. The canonical identities for trace relation and chiral symmetry that should be satisfied by the above vertex functions [4] are as follows,

$$\Delta_{\mu\nu}^5(0, p, -p) = (2 - p\partial_p)\Pi_{\mu\nu}^5(p, -p), \quad (9)$$

$$\Delta_\nu^5(0, p, -p) = (2 - p\partial_p)\Pi_\nu^5(p, -p), \quad (10)$$

$$\Delta^5(0, p, -p) = (2 - p\partial_p)\Pi^5(p, -p); \quad (11)$$

$$-ip^\mu \Delta_{\mu\nu}^5(0, p, -p) = \Delta_\nu^5(0, p, -p) + \Pi_\nu^5(p, -p), \quad (12)$$

$$ip^\nu \Delta_\nu^5(0, p, -p) = \Delta^5(0, p, -p) + \Pi^5(p, -p) + \Pi^{\theta\sigma}(0, 0), \quad (13)$$

$$-ip^\mu \Pi_{\mu\nu}^5(p, -p) = \Pi_\nu^5(p, -p), \quad (14)$$

$$ip^\nu \Pi_\nu^5(p, -p) = \Pi^5(p, -p) + \langle\sigma\rangle. \quad (15)$$

The first three are canonical trace identities and the rest are canonical chiral Ward identities.

After some calculations in dimensional regularization we obtain the following one-loop results for the interested objects:

$$\begin{aligned} \Pi_{\mu\nu}^5(p, -p) &= \frac{2g_{\mu\nu}}{(4\pi)^2} \{p^2[\Delta_0 - \Gamma(\epsilon) + 2 \int_0^1 dx(x^2 - x)(\ln \frac{D}{4\pi\mu^2} - \Gamma(\epsilon) - 1)] \\ &\quad - 2m^2[\Delta_0 + \ln \frac{m^2}{4\pi\mu^2} - 2\Gamma(\epsilon)]\} \\ &\quad + \frac{p_\mu p_\nu}{4\pi^2} \{2 \int_0^1 dx(1-x)^2[\ln \frac{D}{4\pi\mu^2} - \Gamma(\epsilon)] + \Gamma(\epsilon) - \Delta_0\}, \end{aligned} \quad (16)$$

$$\Delta_{\mu\nu}^5(0, p, -p) = \frac{g_{\mu\nu}m^2}{2\pi^2} \{2\Gamma(\epsilon) - \ln \frac{m^2}{4\pi\mu^2} - \Delta_0 + \frac{p^2/2 - 2m^2}{\Delta}\} + \frac{m^2 p_\mu p_\nu}{\pi^2 p^2} (1 - \frac{m^2}{\Delta}); \quad (17)$$

$$\Pi_\nu^5(p, -p) = \frac{im^2 p_\nu}{2\pi^2} (\Delta_0 - \Gamma(\epsilon)), \quad (18)$$

$$\Delta_\nu^{5,\epsilon}(0, p, -p) = \frac{im^2 p_\nu}{2\pi^2} (\Delta_0 - \Gamma(\epsilon) + \frac{2m^2}{\Delta}); \quad (19)$$

$$\Pi^5(p, -p) = \frac{m^2}{2\pi^2} \{2m^2(\ln \frac{m^2}{4\pi\mu^2} - \Gamma(\epsilon) - 1) - p^2(\Delta_0 - \Gamma(\epsilon))\}, \quad (20)$$

$$\Delta^5(0, p, -p) = \frac{m^4}{\pi^2} \{2(\ln \frac{m^2}{4\pi\mu^2} - \Gamma(\epsilon)) - \frac{p^2}{\Delta}\}; \quad (21)$$

$$\langle\sigma\rangle = \frac{m^4}{\pi^2} (\Gamma(\epsilon) + 1 - \ln \frac{m^2}{4\pi\mu^2}), \quad (22)$$

$$\Pi^{\theta\sigma}(0, 0) = \frac{3m^4}{\pi^2} (\Gamma(\epsilon) + 1/3 - \ln \frac{m^2}{4\pi\mu^2}), \quad (23)$$

with $D = m^2 + p^2(x^2 - x)$, $\Delta_0 = \int_0^1 dx \ln \frac{D}{4\pi\mu^2}$, $\frac{1}{\Delta} = \int_0^1 \frac{dx}{D}$. Now we could check them with our results given above. Inserting these functions into Eqs. (9, 10, 11, 12, 13, 14, 15), we find that all the chiral identities are valid for the one-loop functions calculated above in dimensional regularization, but all the trace identities are violated, namely, Eq.s(9, 10, 11) are modified as follows,

$$\Delta_{\mu\nu}^5(0, p, -p) = (2 - p\partial_p)\Pi_{\mu\nu}^5(p, -p) + \frac{1}{6\pi^2}(g_{\mu\nu}p^2 - p_\mu p_\nu) - \frac{m^2}{\pi^2}g_{\mu\nu}, \quad (24)$$

$$\Delta_\nu^5(0, p, -p) = (2 - p\partial_p)\Pi_\nu^5(p, -p) + \frac{im^2}{\pi^2}p_\nu, \quad (25)$$

$$\Delta^5(0, p, -p) = (2 - p\partial_p)\Pi^5(p, -p) - \frac{m^2}{\pi^2}p^2 + \frac{2m^4}{\pi^2} \quad (26)$$

Now anomalies appear in all the three trace identities. That means in dimensional regularization, the chiral Ward identities are preserved in these three- and two-point functions for

the axial operators considered, but the trace identities are quantum mechanically violated. The above results are obtained without use of the chiral Ward identities, unlike the procedures taken in Ref. [4]. To compare our results with previous ones and to check if the chiral Ward identities are consistent with these anomalous trace identities, we follow the procedures of Ref. [4]. We should also note that these anomalous identities still hold even after minimal-like subtraction, that is, they are valid for both unrenormalized and renormalized vertex functions.

That is, we apply the relations Eqs.(12, 13, 14, 15) to the first of the anomalous equation to derive the other two. Noting that

$$\frac{2m^4}{\pi^2} = 3\langle\sigma\rangle - \Pi^{\theta\sigma}(0,0), \quad (27)$$

we arrive at the following form of the anomalous trace identities:

$$\Delta_{\mu\nu}^5(0, p, -p) = (2 - p\partial_p)\Pi_{\mu\nu}^5(p, -p) + \frac{1}{6\pi^2}(g_{\mu\nu}p^2 - p_\mu p_\nu) - \frac{m^2}{\pi^2}g_{\mu\nu}, \quad (28)$$

$$\Delta_\nu^5(0, p, -p) = (2 - p\partial_p)\Pi_\nu^5(p, -p) + \frac{im^2}{\pi^2}p_\nu, \quad (29)$$

$$\Delta^5(0, p, -p) = (2 - p\partial_p)\Pi^5(p, -p) - \frac{m^2}{\pi^2}p^2 + 3\langle\sigma\rangle - \Pi^{\theta\sigma}(0,0). \quad (30)$$

Now we find complete agreement between the two approaches, since Eqs.(28, 29, 30) are exactly the same as Eqs.(24, 25, 26), due to the relation given Eq.(27).

However, comparing Eqs.(28, 29, 30) or Eqs.(24, 25, 26) with those in Ref. [4], we find two disagreements: (1) In Ref. [4], the numerical coefficient of the anomaly term ($g_{\mu\nu}p^2 - p_\mu p_\nu$) is $\frac{1}{8\pi^2}$ while here in Eq.(28) it is $\frac{1}{6\pi^2}$; (2) In Ref. [4] the last two terms in Eq.(30) (or the $\sim m^4$ term in Eq.(26)) were missing.

The most interesting anomalous identity is Eq.(30). From the trace identity perspective, both $-\frac{m^2}{\pi^2}p^2$ and $3\langle\sigma\rangle - \Pi^{\theta\sigma}(0,0)$ are anomalies. However, the latter is required by and explicable within the chiral Ward identities and its existence is independent of regularization or short distance physics, thus we find an interesting phenomenon: *the canonical terms in chiral identity become anomalies in trace identity*. To our knowledge, this phenomenon has not yet been reported in field theory and high energy physics literature. It is known that in supersymmetric field theories, the gauge field components from the trace anomalies ($\sim \text{tr}(F^2)$) and from chiral anomalies ($\sim \text{tr}(F\bar{F})$) comprise a supermultiplet and hence share the same coefficient [9], thus chiral 'symmetry' and scale 'symmetry' are closely related in supersymmetric contexts. Here we encountered another phenomenon between the trace identities and chiral identities, in the axial scalar sectors as the 'current' density j_5 couples to axial scalar fields. The deeper implications of this interesting finding is not clear for us yet. We refrain from making speculations about it before further investigation is carried out. Whilst we believe that it is worthwhile to pay attention to this phenomenon. Lastly we mention that this phenomenon is independent of regularization scheme, for we have also calculated all the vertex functions in a general parametrization of regularization schemes and reobtained Eq.(30), for details see [10].

In summary, we have investigated the trace identities and chiral identities for certain vertex functions of axial operators, by one loop calculations in dimensional regularization

directly. Some disagreements with previous publications and an interesting phenomenon were found.

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